Math 131A-3: Homework 6

Due: November 8, 2013

1. Do problems 17.3 (a),(c), 17.10, 17.12, 18.4, 18.7, 18.10, 19.1(a),(c),(f),(g), 19.2(b), 19.4 in Ross.

2. *The stars over Babylon.* For each rational number $r \in (0,1]$, write $r = \frac{p}{q}$ where $p, q \in \mathbb{N}$ are natural numbers with no common factors. Then consider the following function on $[0,1]$:

$$f(x) = \begin{cases} \frac{1}{q} & x = \frac{p}{q} \text{ is rational} \\ 0 & x \text{ is irrational.} \end{cases}$$

We claim that f is discontinuous at every rational number in $(0,1]$ and continuous at every rational.

- To show f is discontinuous at every rational, we first need to show that every interval (a,b) in $\mathbb{R}$ contains an irrational number. (Which hasn't been touched on in lecture, although it may have been mentioned in section.) So we'll do that quickly: first, show that for r rational, $r + \sqrt{2}$ is irrational. Then use the fact that any interval contains a rational to show that every interval contains a number of the form $r + \sqrt{2}$.
- Discontinuity at each rational. Let $x_0 \in (0,1]$ such that x_0 is rational. For $n \in \mathbb{N}$, pick x_n an irrational in $(x_0 - \frac{1}{n}, x_0) \cap (0,1]$. Use this sequence to show f is discontinuous at x_0.
- Continuity at each irrational. Let $x_0 \in (0,1]$ such that x_0 is irrational. Let N be a natural number. Let

$$\delta_N = \min\{|x_0 - \frac{i}{n}| : 0 \le i \le n \le N,\ i, n \in \mathbb{N}\}.$$

Observe that because $x_0 \neq \frac{i}{n}$ for any $\frac{i}{n}$, $\delta_N > 0$. Prove that for $x \in (0,1]$, if $|x - x_0| < \delta_N$, then $|f(x) - f(x_0)| < \frac{1}{N}$. Conclude that f is continuous at x_0.

This example helps demonstrate that our intuition for what continuity should "look like" on a graph is in general insufficiently subtle.